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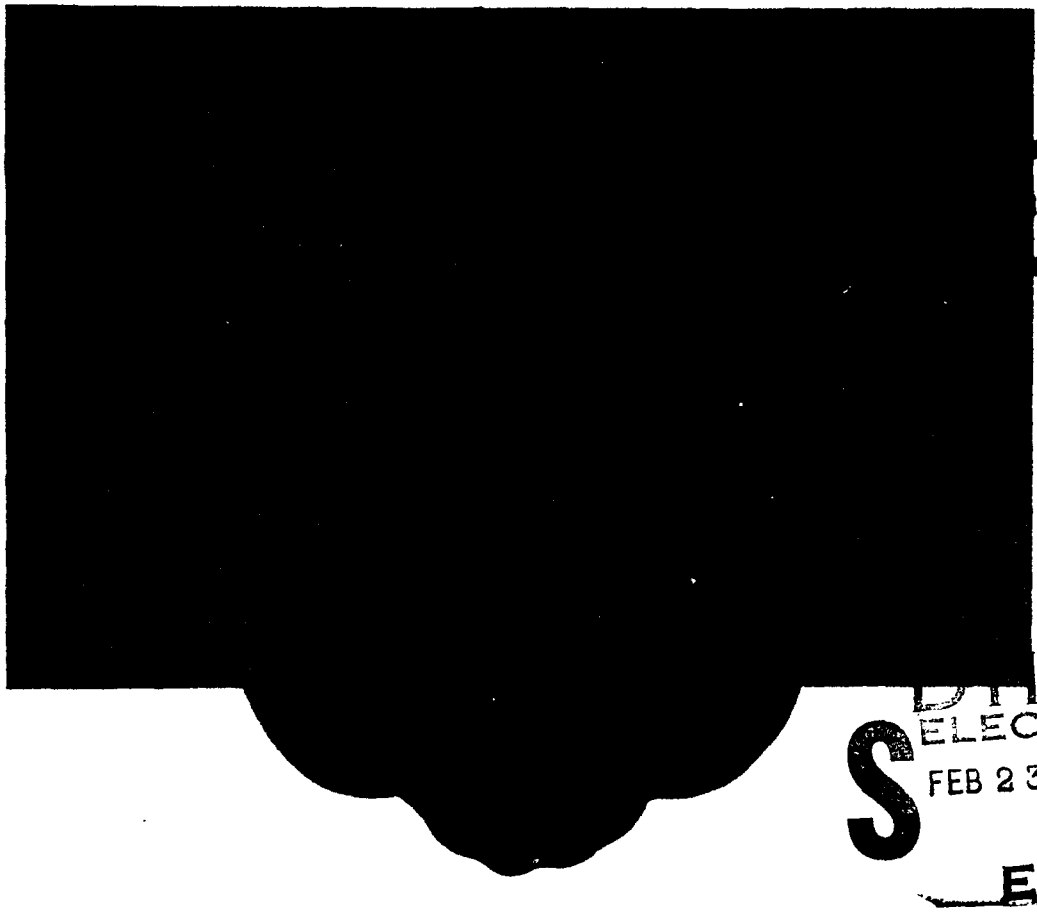






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On a Class of Multivariate Negative Binomial Distributions

by

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ON A CLASS OF MULTIVARIATE NEGATIVE BINOMIAL DISTRIBUTIONS

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ABSTRACT

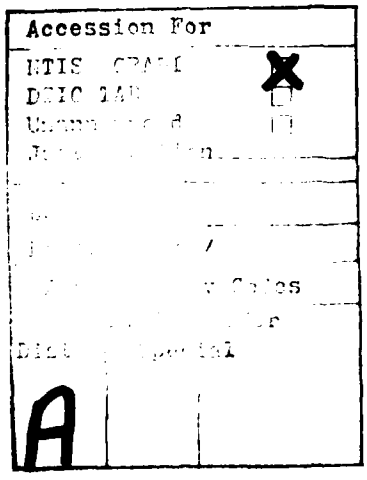
Doss (*J. Multiv. Anal.* 9 (1979), 460-464) considered distributions on $\{0,1,2,\ldots\}^m$, whose probability generating functions were of the form

$$G(s_1,\ldots,s_m) = \{a_0 + \sum_i a_i s_i + \sum_{i<j} a_{ij} s_i s_j + \ldots + a_{1\ldots m} s_1 \ldots s_m\}^{-\gamma}$$

where $\gamma > 0$. Whenever G is a probability generating function, except in degenerate cases, the corresponding distribution is an m-variate distribution with negative binomial distributions for its univariate marginals. Apart from a few comments in the bivariate case, Doss gave no discussion of either necessary or sufficient conditions on the a's for G to be a probability generating function. This paper discusses the known sufficient conditions and obtains various necessary conditions. It is shown that the region of convergence of the associated power series expansion about the origin plays a key role. Attention is concentrated on the bivariate case: even here it seems difficult to obtain conditions which are both necessary and sufficient.

AMS 1970 Subject Classifications: Primary 62H05: Secondary 60E05.

Key Words and Phrases: Multivariate geometric, multivariate negative binomial, characterization of probability distributions.



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1. INTRODUCTION

The negative binomial distribution arises in probability as the distribution of the waiting time to achieve a specified number of successes in a sequence of Bernoulli trials. In addition, it has been widely used in statistics as a model for a variety of data involving counts. Johnson and Kotz ((1969), Chapter 5) and Douglas (1979) give detailed discussion of this distribution and its properties.

Multivariate analogues of the negative binomial distribution are of interest as joint distributions of waiting times in Bernoulli trials and for modelling data involving pairs or m-tuples of possibly dependent counts.

This paper considers m-variate distributions concentrated on $\mathbb{Z}_+^m \equiv (\mathbb{Z}_+)^m$ where $\mathbb{Z}_+ = \{0,1,2,\ldots\}$ and having negative binomial univariate marginal distributions. Such distributions will be called *multivariate/m-variate negative binomial distributions*. This terminology is consistent with, for example, that of Johnson and Kotz (1969) and Lancaster (1969), where multivariate x distributions have, by definition, univariate marginals of type x. Not all authors have, however, adopted this usage. For example, Doss (1979) employs a restricted definition which calls a distribution m-variate negative binomial iff it has a joint probability generating function (p.g. fn.) of the form

$$G(s_1,\ldots,s_m) = \{H(s_1,\ldots,s_m)\}^{-\gamma} , \qquad (1)$$

where $\gamma > 0$ is not necessarily an integer, and H is of the form

$$H(s_1,\ldots,s_m) = a_0 + \sum_{1\le i\le m} a_i s_i + \sum_{1\le i<j\le m} a_{ij}s_i s_j + \ldots$$

$$\ldots + \sum_{1\le i_1<\ldots<i_r\le m} a_{i_1\ldots i_r} s_{i_1}\ldots s_{i_r} + \ldots$$

$$\ldots + a_{12\ldots m} s_1 s_2 \ldots s_m \ . \qquad (2)$$

If (1) is the p.g. fn. of an m-variate distribution which is non-degenerate (in the sense that it is not concentrated on some subspace of $\mathbb{R}^m$ having dimension strictly less than m), then all of its univariate marginal distributions are negative binomial. In the case m = 2, distributions of this general form had been studied previously by Wiid (1958) and Edwards and Gurland (1961). (See also the discussion towards the end of §2 and early in §4 of the present paper.)

It is not hard to see that distributions of this form do not, for any fixed m, exhaust the class of all m-variate negative binomial distributions. Several types of counterexample can be constructed. One type can be obtained by appropriate shifts of probability mass starting from a distribution with p.g. fn. of the form (1). Consider the case m = 2. It is enough to start from a p.g. fn. of the form

$$G(s_1,s_2) = G_1(s_1)G_2(s_2) \ , \qquad (3)$$

where $G_i(s) = (a_i + b_i s)^{-\gamma}$, i = 1,2, are (univariate) negative binomial p.g. fns. with the same index $\gamma > 0$. Then for ε suitably small, one can always shift mass ε away from each of the points (0,0),(1,1) and place a further mass of ε at each of (1,0),(0,1). Such a shift of mass clearly preserves the marginal distributions but not the independence which is implicit in (3). Assuming $\varepsilon > 0$, it is easy to see that the new

distribution exhibits negative covariance; alternatively allowing $\varepsilon < 0$ would introduce positive covariance. The p.g. fn. of this distribution is easily written down and is clearly not of the form (1) with $m = 2$.

Another type of counterexample can be constructed by mixing. Let G_1, G_2 be any two distinct bivariate p.g. fns. of the form (1) with corresponding marginals identical. Then

$$G(s_1, s_2) = \tfrac{1}{2}[G_1(s_1, s_2) + G_2(s_1, s_2)]$$

is a bivariate p.g. fn. whose marginal p.g. fns. are exactly those of G_1 and G_2 but whose form cannot be given by (1).

A further type of counterexample can be provided within the family of distributions that variously bears the names of Eyraud, Farlie, Gumbel, and Morgenstern: see, for example, Cambanis (1977).

Nevertheless, as evidenced at least by the results of Theorem 1 and Theorem 2 of Doss's paper, multivariate negative binomial distributions with p.g. fns. of the form (1) do have a special interest. Doss's main result (Theorem 1) can be expressed in our terminology as follows:

PROPOSITION. *Within the class of all m-variate negative binomial distributions, the subclass consisting of all those which are m-variate linear exponential distributions coincides with the subclass of those whose p.g. fns. are of the form* (1).

The statistical importance of this characterization of distributions whose p.g. fns. are of the form (1) flows from the importance in statistical inference of linear exponential distributions; families of such distributions are the mathematically and statistically "nice" families of distributions: see, for example, Johansen (1979).

In spite of the characterization expressed by the above proposition, there is apparently no known characterization of the class of a's for which (1), with H given by (2), is a p.g. fn. In the next section we discuss known sufficient conditions through a relationship with infinitely divisible multivariate Poisson distributions. Section 3 attacks the problem of obtaining necessary conditions: the approach involves consideration of the convergence behaviour of power series in several variables and necessary conditions for such series to have non-negative coefficients. Such convergence questions do not seem to have received attention in the probability literature. The basic technique for obtaining necessary conditions for the non-negativity of the coefficients was used first in [19] and subsequently in [25] and [26]. We establish by this means several sets of necessary conditions, one of which, (9), may well be sufficient. Details are presented mainly in the bivariate case. The final section mentions a possible characterization of p.g. fns. of the form (1) by infinite divisibility: an analogue of the characterization for multivariate Poisson distributions that is quoted in Section 2. In addition, it discusses a special subclass of p.g. fns. of the form (1) that has been somewhat more extensively studied in the literature.

2. SUFFICIENT CONDITIONS

In order for a function G of the form (1), with H given by (2), to be the p.g. fn. of a distribution on $\mathbb{Z}_+^m$, it is sufficient that a_0 be positive and that all the other a's be non-positive. If degeneracy is to be avoided, some further restrictions are needed: we will comment later on these in the bivariate case. The sufficiency is readily seen by

considering an appropriate mixture of a particular type of multivariate Poisson distribution.

It is known (cf. Patil and Joshi (1968), p. 82; Johnson and Kotz (1969), p. 300) that

$$\exp\{H(s_1,\ldots,s_m)\}\ , \tag{4}$$

where H is given by (2) with $a_0 > 0$ and all the a's other than a_0 non-negative, is the p.g. fn. of a multivariate distribution with Poisson marginals. In fact a_0, which is determined by the condition that the probabilities must sum to one, is given by $-a_0$ = the sum of all the other a's. The general form of such distributions was apparently first noted by Loève ((1950), p. 84) and independently by Krishnamoorthy (1951), the bivariate case having been considered earlier by Campbell (1934). Dwass and Teicher (1957) have shown that the class of all distributions with p.g. fns. of the form (4) is exactly the class of all infinitely divisible m-variate distributions with Poisson marginals.

Suppose now that for each multi-index $\underset{\sim}{k}$, $a_{\underset{\sim}{k}} = \lambda\alpha_{\underset{\sim}{k}}$ where λ and $\alpha_{\underset{\sim}{k}}$ for $\underset{\sim}{k} \neq 0$ are non-negative. Denote the resultant p.g. fn., obtained from (4), by $K_\lambda(s_1,\ldots,s_m)$. Then for any probability distribution F on $[0,\infty)$, K defined by

$$K(s_1,\ldots,s_m) = \int_0^\infty K_\lambda(s_1,\ldots,s_m)F(d\lambda)$$

is another m-variate p.g. fn. If F is the gamma distribution with moment generating function $M(t) = (1-t)^{-\gamma}$ where $\gamma > 0$, then taking account of the condition $a_0 = -\sum a_{\underset{\sim}{k}}$, one finds that K is of the form (1) with H given by (2), except that now $a_{\underset{\sim}{k}} = -\alpha_{\underset{\sim}{k}}$ for each multi-index $\underset{\sim}{k} \neq 0$ and

$a_0 = 1 - \alpha_0$. Thus K is of the form (1) with $a_0 > 0$ and $a_{\underset{\sim}{k}} \leq 0$ for $\underset{\sim}{k} \neq 0$. This proves the assertion made at the beginning of this section since any such K can be obtained by the process just described.

In the bivariate case this construction appears to have been used in their studies of accident proneness, first by Arbous and Kerrich (1951) in the case of independent Poissons, and then by Edwards and Gurland (1961) in the case of an arbitrary infinitely divisible bivariate Poisson distribution. The m-variate case starting from independent Poissons was considered in a similar context by Bates and Neyman (1952).

One might suspect that some other mixing process could produce a wider class of p.g. fns. of the form (1). It is not hard to check that a mixing process similar to the above but with $a_{\underset{\sim}{k}} = \lambda_{\underset{\sim}{k}} \alpha_{\underset{\sim}{k}}$, $\underset{\sim}{k} \neq 0$ and the $\lambda_{\underset{\sim}{k}}$'s iid F would not result in a p.g. fn. of the form (1) starting from the general infinitely divisible m-variate Poisson form (4). However, if attention is restricted to the case of independent Poissons and the mixture then performed according to the modified scheme, the resultant p.g. fn. is a product of univariate negative binomial p.g. fns. This does result in a wider class of p.g. fns. since, as we shall see in the next section, some of the $a_{\underset{\sim}{k}}$'s must now be positive.

3. NECESSARY CONDITIONS

In this section, so as to simplify the discussion, we will initially restrict attention to the bivariate case of (1), i.e. to the form

$$G(s_1, s_2) = \{a_0 + a_1 s_1 + a_2 s_2 + a_{12} s_1 s_2\}^{-\gamma} \tag{1b}$$

where $\gamma > 0$. The previously observed sufficient condition for (1) to be

a p.g. fn. reduces to

$$a_0 > 0 \text{ and } a_1, a_2, a_{12} \leq 0 \ . \qquad (S)$$

In fact, if (1b) is to be a non-degenerate bivariate p.g. fn., we must have, in addition, at least one of the inequalities $a_{12} \neq 0$, $a_1 a_2 \neq 0$ satisfied, and then each marginal will be a negative binomial distribution. If we are not concerned about possible degeneracy but only with when a function of the form (1b) is a p.g. fn., albeit perhaps degenerate, then such additional conditions need not be considered.

That the conditions (S) are not also necessary was observed by Doss (1979). He pointed out that the case of independence is not covered by (S): in this case it is easy to check (cf. Case II later in this section) that necessarily, except in degenerate situations,

$$a_0, a_{12} > 0 \text{ and } a_1, a_2 < 0 \ .$$

Having made this point, Doss avoided further discussion of which a's were permissible by allowing in his Definition 1 any real a's consistent with (1b), or (1) in the general case, being a p.g. fn. For his purposes, this was adequate: it did not appear to affect the approach to his characterization results. However, the problem of trying to find what a's are permissible does seem worthy of further consideration. One might even regard a characterization such as Doss's Theorem 1 as relatively powerless unless this problem is solved. Nevertheless, as the remainder of this section will indicate, the problem of actually characterizing the class of permissible a's appears to be a difficult one.

In order that a function of the form (1) with H given by (2) be a

p.g. fn., it is necessary that:

(a) it possess a power series expansion about the origin, (absolutely) convergent for $(s_1,\ldots,s_m)$ in some non-trivial region C;

(b) this series be such that *all* its coefficients are non-negative; and (c) $G(1,\ldots,1) = 1$.

Restricting still to the bivariate case, we examine first the convergence behaviour.

In general, when $m = 1$ we know that the power series defining any p.g. fn. will have a non-trivial radius of convergence $R \geq 1$. For some distributions we may have $R = +\infty$, in which case the series represents an entire function. For the univariate negative binomial distribution with p.g. fn. $(a+bs)^{-\gamma}$, where $a,\gamma > 0$ and $a + b = 1$, we have $R = |a/b| < +\infty$.

When $m > 1$, an even when $m = 2$, the situation is usually much more complicated. Corresponding to any power series of the form

$$\sum_{m,n \in \mathbb{Z}_+} c_{mn} s_1^m s_2^n , \tag{5}$$

there is a set, Γ, of pairs (r_1,r_2) of so-called associated radii of convergence (cf. Behnke and Thullen (1970), §III.1; Fuks (1963), p. 46), any pair being such that the series converges absolutely for $|s_1| < r_1$, $|s_2| < r_2$ and diverges for $|s_1| > r_1$, $|s_2| > r_2$. One can consider also maximal radii of convergence where these are defined as upper limits: $R_i = \sup\{r_i : (r_1,r_2) \in \Gamma\}$, $i = 1,2$. It is known that Γ is a curve which can be described by either of the equations $r_2 = \phi(r_1)$ or $r_1 = \psi(r_2)$, say, where ϕ and ψ are non-increasing continuous functions ([5], p. 50; [12], p. 46). Thus Γ is a curve of rather special structure which describes the boundary of the region C of convergence. Further, the

image of C under the log map $(s_1 \to \ln s_1\,,\ s_2 \to \ln s_2)$ is a convex region ([3], p. 50).

In the p.g. fn. case, i.e. when $c_{mn} \geq 0$ for all $m,n \in \mathbb{Z}_+$ and $\sum c_{mn} = 1$, it is natural to restrict attention at least to $s_1 \geq 0$ and $s_2 \geq 0$. Moreover, for a p.g. fn. it is always true that $(1,1) \in C$, or equivalently that C contains the unit square $[0,1]\times[0,1]$, and so for many purposes one can further restrict to this latter domain. For most applications of bivariate p.g. fns., this domain suffices: indeed, we have been unable to find any text on probability theory which discusses questions of convergence since they all restrict attention simply to the unit square. However, as will become evident soon, this does not suffice for our present purposes: the associated region C of convergence, or equivalently the curve Γ, is intimately connected with the problem of deciding when a convergent power series (about the origin) has non-negative coefficients i.e. with (b) above.

We will return to discuss the convergence behaviour of the series associated with (1b), but first, partly in order to narrow the number of special cases that need to be considered when discussing convergence, we investigate the non-negativity of the coefficients of the series.

The paper by Kemp and Kemp (1965) considered functions G of the form

$$G(s) = \exp\{a_1(s-1) + a_2(s^2-1)\}\ . \tag{6}$$

An obvious sufficient condition for this to be a p.g. fn. is $a_1, a_2 \geq 0$, since then G is a particular compound Poisson distribution (in the sense of Feller (1968) and others). By means of an elegant but quite elementary technique, Kemp and Kemp showed that the condition $a_1, a_2 \geq 0$

is also necessary. For reasons made clear in their paper, they called the distribution corresponding to (6) the Hermite distribution. A similar technique was used to advantage by Milne (1971) and Milne and Westcott (1972) to characterize certain multivariate p.g. fns. that arose in the context of point processes: in the latter paper, the distributions involved were certain multivariate analogues of the Hermite distribution; in the former, it was shown that a function of the form (4), with H given by (2), is a p.g. fn. iff all a's other than a_0 are non-negative (as usual, a_0 is determined by the normalization), the necessity of this condition having not been pointed out previously. In order to facilitate a related approach and an appreciation of its limitations in the context of multivariate negative binomial distributions, we describe the basic technique in the original Kemp and Kemp setting.

Since (6) is an entire function for all a_1, a_2, its power series expansion about $s = 0$ must converge for all (real) s, and because of the exponential form of G, we must have $G(s) > 0$ whenever $s \geq 0$. Hence we can consider $\ell n\, G(s)$. If the coefficients in the power series are to be all non-negative, then both $G(s)$ and $\ell n\, G(s)$ must be non-decreasing functions of s when s is non-negative. It follows that $d/ds\ \ell n\, G(s) \geq 0$, i.e. that

$$a_1 + 2a_2 s \geq 0 \qquad (7)$$

for all $s \geq 0$. Putting $s = 0$ shows that a_1 must be non-negative, while letting $s \to \infty$ yields $a_2 \geq 0$. Thus $a_1, a_2 \geq 0$ is a necessary condition for (6) to be a p.g. fn.

The critical properties needed to obtain this result were:

(i) that $s \geq 0 \Rightarrow G(s) > 0$;

and (ii) that one could let $s \to \infty$ in (7).

Property (i) was a consequence of the (exponential) form of G while property (ii) flowed from the fact that G was entire, i.e. an analytic function for all s. In the cases considered by the other authors mentioned above, the technique generalized straightforwardly: in particular, the functions involved were still entire functions albeit of several variables.

Now consider G given by (1b) with $\gamma > 0$ and suppose that this has a power series expansion about (0,0) with non-negative coefficients. Since only non-negative powers of s_1 and s_2 appear we must have $a_0 > 0$, and this ensures that $G(s_1,s_2) > 0$ for any (s_1,s_2) in the region C of convergence associated with this series. From the non-negativity of the coefficients in the expansion, we deduce that $G(s_1,s_2)$ is a non-decreasing function of each of its arguments, and hence that $\ell n\, G(s_1,s_2)$ is similarly non-decreasing in each of its arguments when these are non-negative. Thus we must have

$$\frac{\partial}{\partial s_i} \ell n\, G(s_1,s_2) \geq 0 , \qquad i = 1,2 ,$$

or equivalently,

$$\begin{aligned} a_1 + a_{12}s_2 &\leq 0 \\ a_2 + a_{12}s_1 &\leq 0 \end{aligned} \tag{8}$$

for all $(s_1,s_2) \in C$. Setting $s_1 = 0 = s_2$ yields $a_1 \leq 0$ and $a_2 \leq 0$. In addition, it follows that

$$a_{12} \leq \min\left(\frac{-a_1}{r_2}, \frac{-a_2}{r_1}\right)$$

for all $(r_1, r_2) \in \Gamma$. Thus the conditions

$$a_0 > 0 \tag{9a}$$

$$a_1 \leq 0\ ,\ a_2 \leq 0 \tag{9b}$$

$$a_{12} \leq \inf_{(r_1,r_2)\in\Gamma} \min\left(\frac{-a_1}{r_2}, \frac{-a_2}{r_1}\right) \tag{9c}$$

are necessary conditions for a function of the form (1b) to have a power series expansion about (0,0) with all its coefficients non-negative. If (1b) is in fact a p.g. fn., then we can add the further condition $G(1,1) = 1$, which implies that

$$a_0 + a_1 + a_2 + a_{12} = 1$$

and hence that $a_0 > 1$.

In general, it seems to be difficult to reduce the condition on a_{12} to something simple. A notable exception arises if at least one of a_1 and a_2 is zero. Then (9c) reduces to $a_{12} \leq 0$. When both a_1 and a_2 are negative, it is possible for a_{12} to be positive: this happens, for example, as we have already observed, in the independence case. Since $(1,1) \in C$ whenever G is a p.g. fn., we can assert that, irrespective of the particular structure of Γ, we must have

$$a_{12} \leq \min(-a_1, -a_2)\ . \tag{10}$$

In the light of the form of C in special cases that will be discussed

shortly, it is clear that (10) is a strictly weaker condition than (9c). Another variant of the condition (9c) on a_{12} can be obtained as follows. Set $R_1' = \sup\{r_1: (r_1,1) \in \Gamma\}$, $R_2' = \sup\{r_2: (1,r_2) \in \Gamma\}$. These are, in fact, the radii of convergence of the respective marginal p.g. fns. Using these and (9c) leads to

$$a_{12} \leq \min\left\{\frac{-a_1}{R_2'}, \frac{-a_2}{R_1'}\right\} , \tag{11}$$

which will, in general, be strictly weaker than (9c) but stronger than (10).

It may well be that the conditions (9) are sufficient as well as necessary, but so far we have been unable to prove or disprove this.

We now classify the relevant convergence behaviour associated with (1b). Clearly, for $s_1,s_2 \geq 0$ the power series expansion of (1b) about the origin will not converge beyond Γ, the closest branch to the origin and in the positive quadrant of the curve given by

$$a_{12}s_1s_2 + a_2s_2 + a_1s_1 + a_0 = 0 . \tag{12}$$

Thus the region of convergence C will be the region between the coordinate axes and the curve Γ. Recall that if G is a p.g. fn., then by (9), $a_0 > 0$ and $a_1,a_2 \leq 0$.

CASE I: $a_{12} = 0$

Here the curve Γ is simply the straight line

$$a_2s_2 + a_1s_1 + a_0 = 0 ,$$

which cuts the respective axes at $-a_0/a_1$ and $-a_0/a_2$. //

From now on, we can and will assume that $a_{12} \neq 0$. Hence (12) can be rewritten as

$$s_1 s_2 + \frac{a_2}{a_{12}} s_2 + \frac{a_1}{a_{12}} s_1 = \frac{-a_0}{a_{12}} ,$$

or equivalently as

$$\left(s_1 + \frac{a_2}{a_{12}}\right)\left(s_2 + \frac{a_1}{a_{12}}\right) = \frac{(a_1 a_2 - a_0 a_{12})}{a_{12}^2} . \tag{13}$$

For our purposes, it is enough to consider cases where the RHS of (13) is non-negative, or equivalently $a_1 a_2/a_0 \geq a_{12}$. For suppose the contrary. Then substituting in the first inequality of (8) would yield

$$a_1 + \frac{a_1 a_2}{a_0} s_2 < 0$$

for all $s_2 > 0$, or equivalently that $s_2 < -a_0/a_2$ for all $s_2 \geq 0$. Since this is a contradiction, we must have $a_1 a_2/a_0 \geq a_{12}$.

CASE II: $a_{12} = a_1 a_2/a_0$

Here F reduces to a pair of straight lines given by the equations

$$s_1 + \frac{a_2}{a_{12}} = 0 , \quad s_2 + \frac{a_1}{a_{12}} = 0 ,$$

and G becomes

$$G(s_1, s_2) = [a_0(1 + a_1/a_0\, s_1)(1 + a_2/a_0\, s_2)]^{-\gamma} .$$

Clearly this is the "independence" case.

From now on, we assume in addition to $a_{12} \neq 0$ that $a_{12} < a_1 a_2/a_0$.

CASE III: $a_1 = 0 = a_2$

Recall that we are then forced by (10) to have $a_{12} < 0$. Then Γ is the upper branch of the curve

$$s_1 s_2 = \frac{-a_0}{a_{12}} ,$$

which is a rectangular hyperbola whose asymptotes are the coordinate axes. In this case, G becomes

$$G(s_1, s_2) = (a_0 + a_{12} s_1 s_2)^{-\gamma} .$$

This reflects a situation of complete dependence: for the corresponding random variables X,Y we have $X = Y$ with probability one, and each has a negative binomial distribution with p.g. fn. $(a_0 + a_{12}\, s)^{-\gamma}$. Notice that here $R_1 = +\infty = R_2$; in the previous two cases, both these quantities were finite.

CASE IV (i): $a_1 = 0$, $a_2 < 0$

Again we must have $a_{12} = 0$. Here Γ is the upper branch of

$$s_1\left(s_2 + \frac{a_1}{a_{12}}\right) = \frac{-a_0}{a_{12}} ,$$

which is a rectangular hyperbola with asymptotes $s_1 = 0$ and $s_2 = -a_1/a_{12}$, the latter quantity being negative. This hyperbola cuts the s_1-axis at $-a_0/a_1$. Thus $R_1 = -a_0/a_1$, but $R_2 = +\infty$. //

CASE IV (ii): $a_1 < 0$, $a_2 = 0$

This follows by symmetry from the previous case. //

CASE V: $a_1 < 0$, $a_2 < 0$

Here again the curve Γ is one of the branches of a rectangular hyperbola, the general equation being given by (13) and the asymptotes by

$$s_1 = \frac{-a_2}{a_{12}} \quad , \quad s_2 = \frac{-a_1}{a_{12}} \quad .$$

Two subcases need to be distinguished:

(i) $a_{12} < 0$. In this case, the asymptotes meet in the negative quadrant, and the relevant branch of the hyperbola is the upper one.

(ii) $0 < a_{12} < a_1 a_2/a_0$. Here the asymptotes meet in the positive quadrant. The lower branch of the hyperbola is clearly the relevant one.

In each of these subcases, the intercepts with the respective coordinate axes, and hence the respective values of R_1 and R_2, are given by $-a_0/a_1$ and $-a_0/a_2$. //

It is instructive to draw the region, C, of convergence for each of the cases and to verify directly that $(1,1) \in C$. With this, we have completed our discussion of the convergence behaviour associated with (1b) in all cases where this is a p.g. fn. It is easy to see that the independence case, Case II, can be viewed as a limit of Case V (ii) as $a_{12} \to a_1 a_2/a_0$. Similarly, Case I may be considered as a limit of

Case V (i) as $a_{12} \to 0$ or of Case V (ii) as $a_{12} \to 0$.

Observe that Case V (ii) is the only one in which condition (9c) will come into play as a necessary condition for the power series expansion of (1b) about (0,0) to have non-negative coefficients. Since in this case the curve Γ is bounded and strictly concave, it seems reasonable to expect that the infinitum in (9c) will be achieved for exactly one pair $(r_1, r_2) \in \Gamma$ for each fixed (a_1, a_2) with $a_1 < 0$ and $a_2 < 0$. It seems difficult, however, to simplify (9c) algebraically, even in this case.

For $m > 2$, the situation will be much more intricate. The analogue of (8) will be provided by the conditions

$$\sum_{\underset{\sim}{k}} a_{\underset{\sim}{k}} \prod_{j \in \underset{\sim}{k} \setminus \{i\}} s_j < 0 , \qquad i = 1,2,\ldots,m , \tag{14}$$

where the sum is over all distinct multi-indices $\underset{\sim}{k} \subseteq \{1,\ldots,m\}$ such that $\{i\} \subseteq \underset{\sim}{k}$, the empty product being interpreted (as usual) as one, and the conditions must be satisfied by all $(s_1,\ldots,s_m) \in C$, the convergence set which is defined by analogy with the bivariate case.

When $m = 3$, the conditions (14) become

$$a_1 + a_{12}s_2 + a_{13}s_3 + a_{123}s_2s_3 < 0 \tag{15a}$$

$$a_2 + a_{12}s_1 + a_{23}s_3 + a_{123}s_1s_3 < 0 \tag{15b}$$

$$a_3 + a_{13}s_1 + a_{23}s_2 + a_{123}s_1s_2 < 0 , \tag{15c}$$

which must be satisfied for all $(s_1,s_2,s_3) \in C$. Putting $s_1 = s_2 = s_3 = 0$ yields $a_1, a_2, a_3 < 0$. Putting $s_3 = 0$ in (15a) and (15b) yields

$$a_1 + a_{12}s_2 < 0 \text{ and } a_2 + a_{12}s_1 < 0 ,$$

which must be satisfied for all s_1, s_2 such that $(s_1, s_2, 0) \in C$. There will be four more conditions similar to this last pair and obtained by setting $s_2 = 0$ in (15a) and (15c) and $s_1 = 0$ in (15b) and (15c). But these nine conditions deduced from (15) do not exhaust all the restrictions contained in (15).

A set of conditions which may be useful even though they are, in general, strictly weaker than (15) can be obtained by using the fact that (1,1,1) must belong to C. We then deduce

$$a_1, a_2, a_3 \leq 0 \tag{16a}$$

$$a_{ij} \leq \min(a_i, -a_j), \qquad i,j = 1,2,3, \qquad i < j \tag{16b}$$

$$a_{123} \leq \min\{-(a_1 + a_{12} + a_{13}), -(a_2 + a_{12} + a_{23}), -(a_3 + a_{13} + a_{23})\}, \tag{16c}$$

which correspond to condition (10) in the bivariate case. Clearly from (16b) we must have $a_{ij} = 0$ as soon as at least one of a_i or a_j is zero. Since then, for example, $a_1 = 0$ forces $a_{12} = 0$ and $a_{13} = 0$, we must by (16c) have, in addition, $a_{123} = 0$.

The convergence behaviour for $m > 2$ will also be complicated. However, just as for $m = 2$ the boundary curve Γ was given by part of a hyperbola or a limiting case thereof, the basic equation being (12), so when $m > 2$ the boundary curve must be given by part of a hyperboloidal surface or a limiting case thereof, where the basic equation is

$$H(s_1, \ldots, s_m) = 0 .$$

To catalogue the behaviour would be messy and not particularly enlightening.

4. CONCLUDING REMARKS

Although the discussion so far has been devoted to obtaining necessary and/or sufficient conditions for (1) to be a p.g. fn., it does seem appropriate to make some remarks about one special subclass of p.g. fns. of the form (1). This is the subclass consisting of those G of the form

$$G(s_1,\ldots,s_m) = \{a_0 + \sum_1^m a_i s_i\}^{-\gamma} \tag{17}$$

where $\gamma > 0$. From §2 and §3, it is clear that such a function is a p.g. fn. iff $a_0 > 0$, $a_1,\ldots,a_m \leq 0$, and $a_0 +\ldots+ a_m = 1$.

Bates and Neyman (1952) were the first to give a systematic study of distributions with p.g. fns. of the form (17), although A. Guldberg (1934) and Arbous and Kerrich (1951) had considered aspects of the bivariate case and S. Guldberg (1935) had noted the general form together with some properties. Lukacs (1973) has given a characterization of the m-variate case with $\gamma = 1$, an m-variate geometric distribution, through a regression property.

Amongst the properties considered by Bates and Neyman was the following, a generalization of which serves, in fact, to characterize p.g. fns. of the form (17) amongst those of the form (1): that for the corresponding random variables $X_1,\ldots,X_m$, the joint distributions of $X_1,X_2,\ldots,X_k$ and $X = X_{k+1} +\ldots+ X_m$ are of the same form for any k such that $1 \leq k < m$. This property of closure under addition is important and serves to explain the appearance of distributions with p.g. fns. of the form (17), but not those of the more general form (1), as the finite-dimensional distributions of point processes whose one-dimensional

distributions are negative binomial: see Gregoire (1980) and Diggle and Milne (1980).

Sibuya et al. (1964) have suggested that since distributions with p.g. fns. of the form (17) bear a relation to negative binomial distributions similar to that between multinomial and binomial distributions, they should be referred to as negative multinomial distributions. Such distributions were the only multivariate negative binomial distributions to be discussed in Johnson and Kotz (1969), which gives a good summary of the various known properties.

Let us mention one other characterization that may be possible for p.g. fns. of the form (1). Recall the Dwass and Teicher (1957) characterization, through infinite divisibility, of p.g. fns. of the form (4). This was quoted earlier in §2. It is clear that whenever a function of the form (1) with H given by (2) is a p.g. fn., it is infinitely divisible: using an obvious notation, we have

$$G_\gamma(s_1,\ldots,s_m) = [G_{\gamma/n}(s_1,\ldots,s_m)]^n$$

for each $n = 1,2,\ldots$. We might conjecture that p.g. fns. of the form (1) are the only possible infinitely divisible p.g. fns. with negative binomial marginals. At the present time, we have been unable to prove or disprove this conjecture.

Finally, we note that Hunter (1969), from a situation with two queues in parallel, derived a functional equation ([16], p. 437, equation (5.7)) for the p.g. fn. of a bivariate distribution, which he showed to have geometric marginal distributions. His attempts to solve the equation failed, and moreover, he showed that it was, in general,

impossible to choose the constants a_0, a_1, a_2, and a_{12} so that a p.g. fn. of the form (1b) would satisfy the equation.

Omission: I am aware of an unpublished manuscript by F. Downton dealing with bivariate exponential and geometric distributions. There may be in this work some material relevant to the present discussions, but unfortunately I am unable to consult it at the moment.

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Doss (*J. Multiv. Anal.* 9 (1979), 460-464) considered distributions on $\{0,1,2,\ldots\}^m$, whose probability generating functions were of the form

$$G(s_1,\ldots,s_m) = \{a_0 + \sum_i a_i s_i + \sum_{i<j} a_{ij} s_i s_j + \ldots + a_{1\ldots m} s_1 \ldots s_m\}^{-\gamma}$$

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20. where $\gamma > 0$. Whenever G is a probability generating function, except in degenerate cases, the corresponding distribution is an m-variate distribution with negative binomial distributions for its univariate marginals. Apart from a few comments in the bivariate case, Doss gave no discussion of either necessary or sufficient conditions on the a's for G to be a probability generating function. This paper discusses the known sufficient conditions and obtains various necessary conditions. It is shown that the region of convergence of the associated power series expansion about the origin plays a key role. Attention is concentrated on the bivariate case: even here it seems difficult to obtain conditions which are both necessary and sufficient.